Wild concealed algebras are not g-tame

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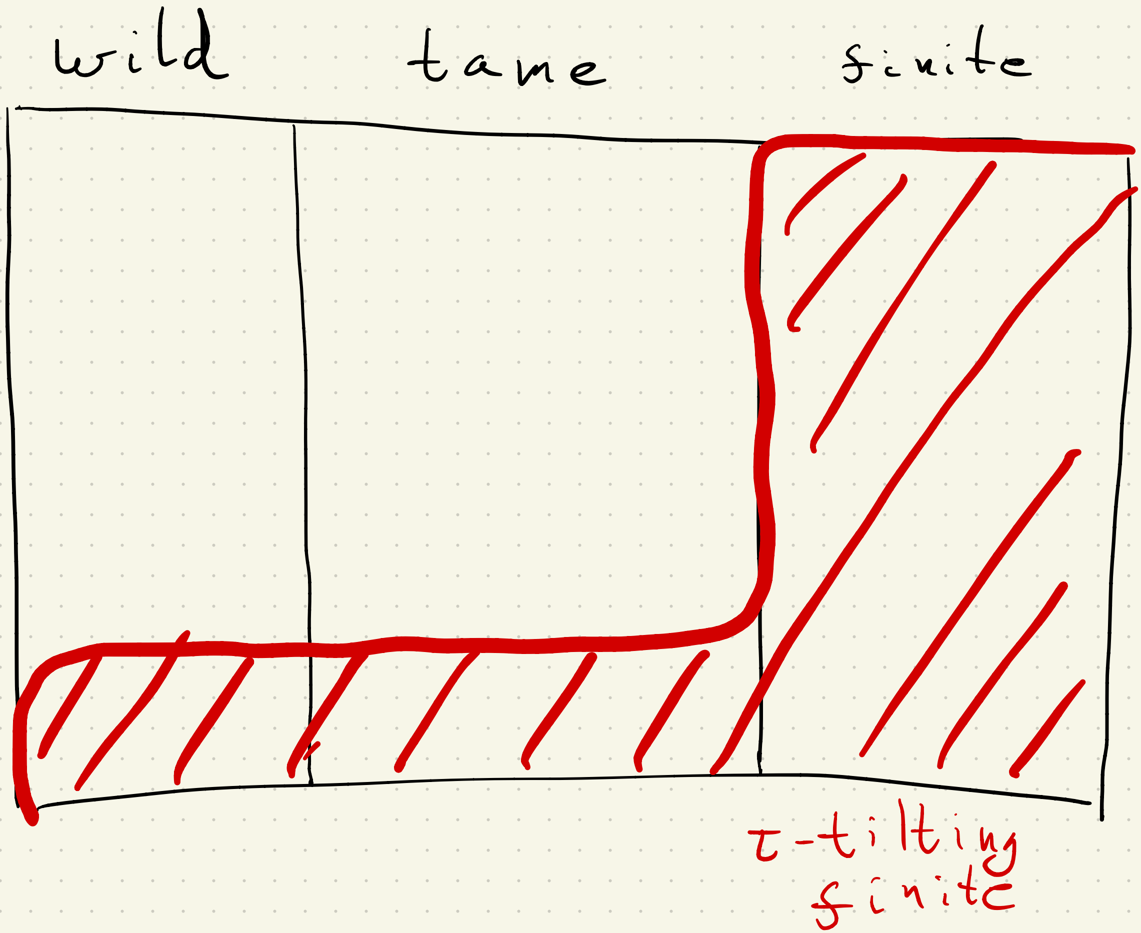
& Endre Rundsveen

ArXiV: 2407.17965

wild	tame	finite


wild
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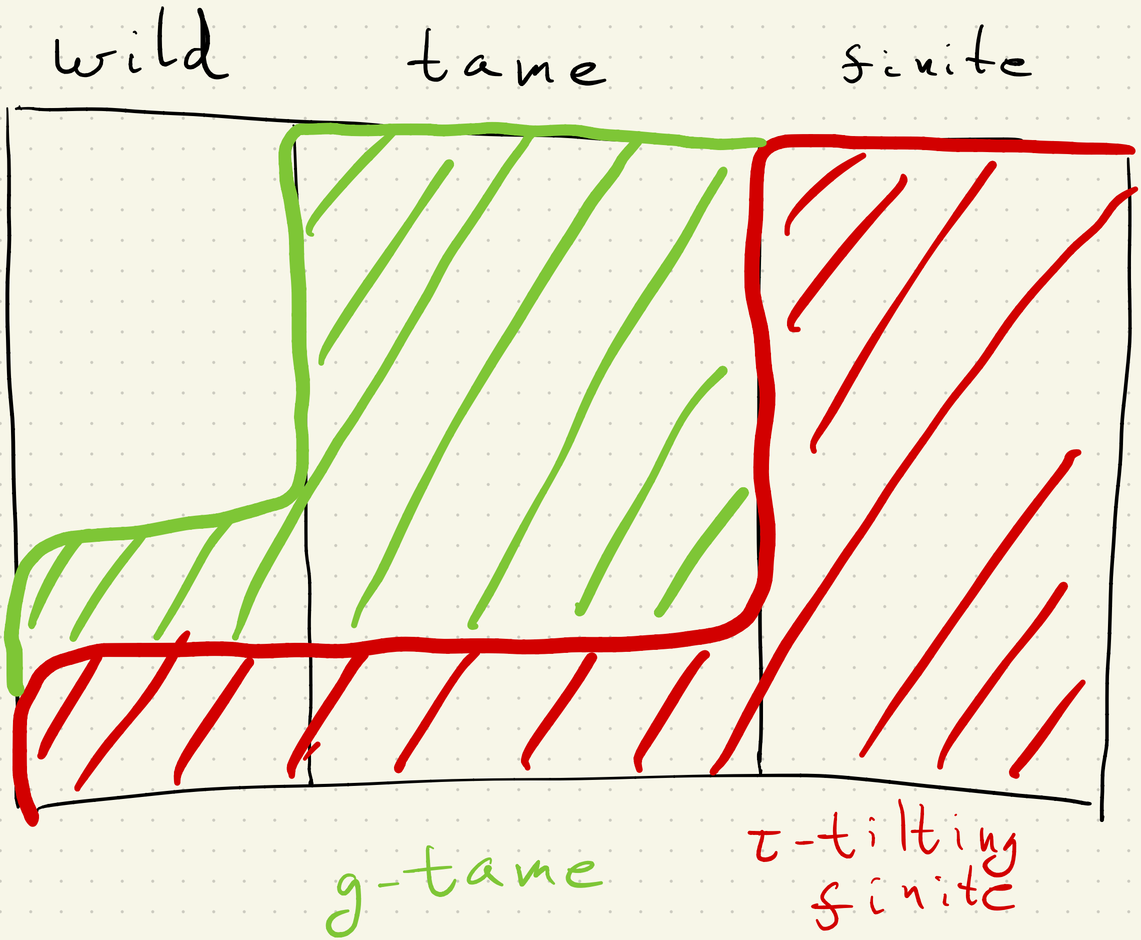
τ-tilting finite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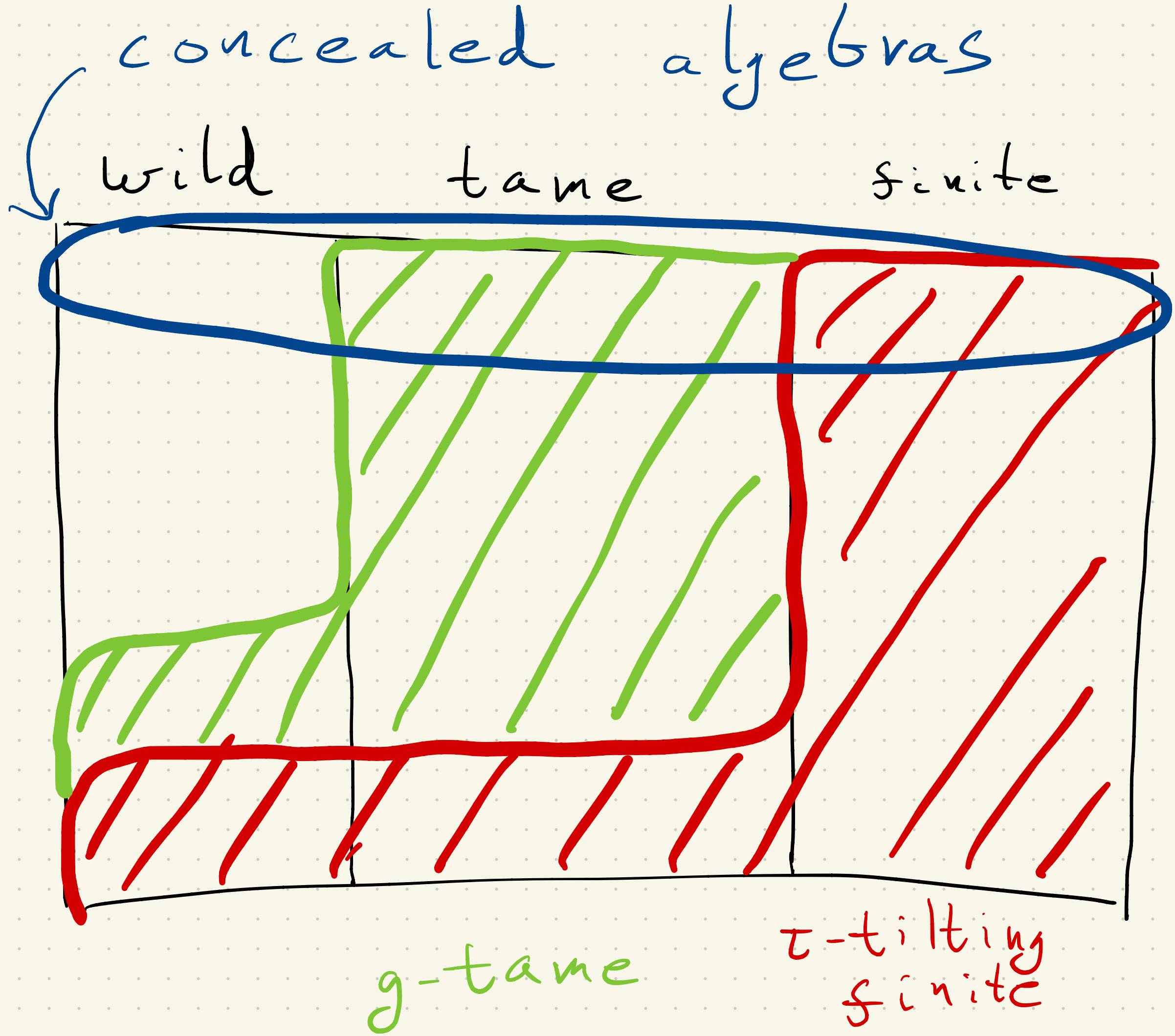
concealed algebras
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τ-rigid pairs

Def: $(M, P) \in \operatorname{mod}\Lambda \times \operatorname{proj}\Lambda$

is a *τ-rigid pair* if

- $\operatorname{Hom}(M, \tau M) = 0$
- $\operatorname{Hom}(P, M) = 0$

It is called *support τ-tilting*

if $|M| + |P| = |\Lambda|$

g-vectors

Def: The g-vector of (M,P) is given by

$$g_{(M,P)} := [P_M^0] - [P_M^1] - [P]$$

where $P_M^1 \longrightarrow P_M^0$ is a minimal presentation of M.

The g-vector fan

Def: $(M,P) = \bigoplus_i U_i$, U_i indecomposable

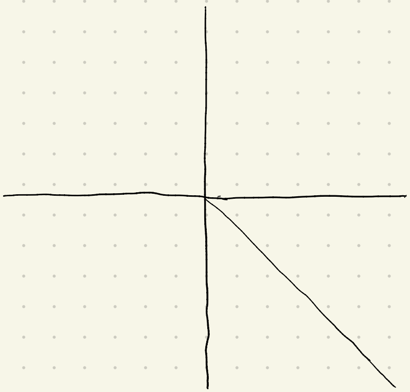
- $C(M,P) = \{ \sum_i a_i g_{U_i} \mid a_i \geq 0 \} \subseteq K_0(\operatorname{proj} \Lambda) \otimes \mathbb{R}$
- $C^+(M,P) = \{ \sum_i a_i g_{U_i} \mid a_i > 0 \} \subseteq K_0(\operatorname{proj} \Lambda) \otimes \mathbb{R}$

The g-vector fan is given by

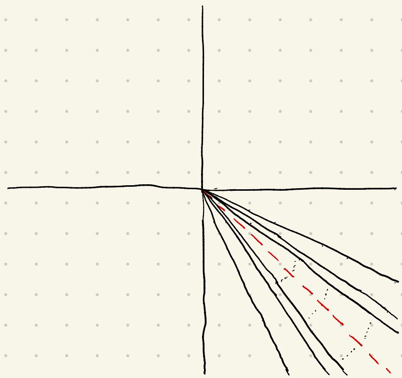
$$\bigcup_{\substack{(M,P) \\ \tau\text{-rigid}}} C(M,P) = \bigcup_{\substack{(M,P) \\ \tau\text{-rigid}}} C^+(M,P)$$

The g-vector fan

$1 \rightarrow 2$



$1 \rightrightarrows 2$



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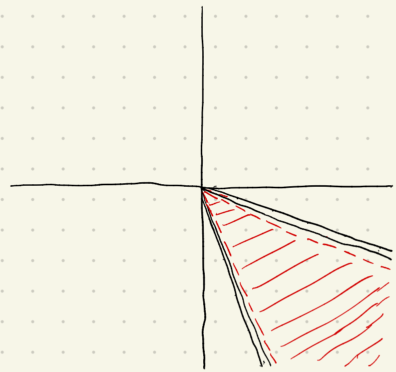

g-tameness

Def: An algebra is g-tame if its g-vector fan is dense in $\mathbb{R}^n$.

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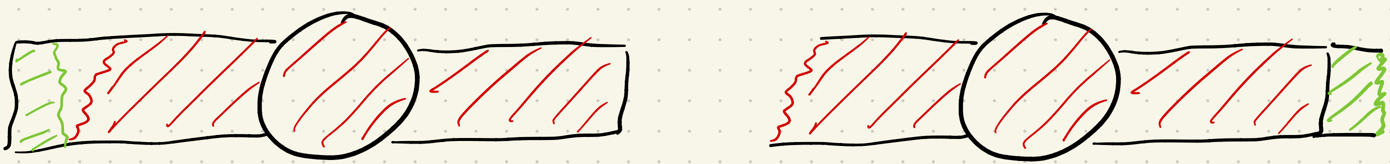
Thm [Plamondon–Yurikusa '23] Tame algebras are g-tame, but the converse is not true.

Concealed algebras

Def: Λ is concealed of type Q if $\Lambda \cong \mathrm{End}_{kQ}(T)$ where T is a postprojective tilting module.

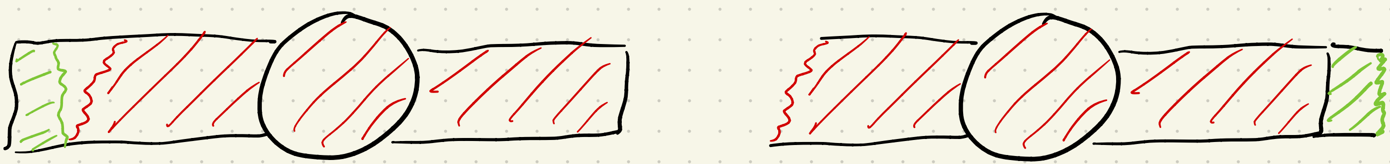


Concealed algebras

$$\mathrm{Hom}_{kQ}(T,-)\colon \mathrm{mod}\,kQ \longrightarrow \mathrm{mod}\,\Lambda$$

$$T^{\perp} \xrightarrow{\cong} \mathrm{Sub}\,DT$$

$\boxed{\text{Maps } g\text{-vectors from } T^{\perp} \text{ by linear isomorphism}}$



Walls and Chambers

Def (Walls):

$$\Theta_M = \left\{ \theta \in K_0(\mathrm{proj}\,\Lambda) \otimes \mathbb{R} \;\middle|\; \begin{array}{ll} \theta(M) = 0 & \\ \theta(X) \geq 0 & X \in \mathrm{Fac}\, M \end{array} \right\}$$

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Thm [Asai '21]

$$K_0(\mathrm{proj}\,\Lambda) \subseteq \bigcup_{\substack{\mathcal{U} \\ s\tau\text{-tilt}}} C^+(\mathcal{U}) \;\sqcup\; \bigcup_{M} \Theta_M$$

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$$\boxed{\text{Remark: } \Lambda \text{ g-tame} \iff \text{Walls nowhere-dense}}$$

Hereditary algebras

$$K_0(\mathrm{mod}\, H) \xrightarrow{\cong} K_0(\mathrm{proj}\, H)$$

$$[M] \longmapsto g_M$$

Euler form: $q_H([M]) = g_M(M)$

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Remark: $q_H([M]) = \dim \operatorname{Hom}(M,M) - \dim \operatorname{Ext}(M,M)$

Hereditary algebras

Thm [Kac '83] H minimally wild,

$x \in \mathbb{Z}^n$, $q_H(x) \leq 1$

$\Rightarrow x$ or $-x = \underline{\dim} M$, M indecomposable

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Prop: $\phi : K_0(\operatorname{proj} H) \xrightarrow{\cong} K_0(\operatorname{proj} \Lambda)$

$q_H(x) < 0$, $x \geq 0$ $\Rightarrow \exists M$ $x \in \Theta_M$

$\phi(x) \in \Theta_{\operatorname{Hom}(T,M)}$

Hyperbolically concealed algebras

Prop: $\phi: K_0(\mathrm{proj}\, H) \xrightarrow{\cong} K_0(\mathrm{proj}\, \Lambda)$

$$q_H(x) < 0,\ x \geq 0 \implies \exists M \quad x \in \Theta_M$$

$$\phi(x) \in \Theta_{\mathrm{Hom}(T,M)}$$

$\implies$ Walls inside $\{x \geq 0 \mid q_H(x) < 0\}$

are preserved when going

from H to Λ.

Hyperbolically concealed algebras

$\Rightarrow$ Walls inside $\{x \geq 0 \mid q_H(x) < 0\}$ are preserved when going from H to Λ.

$\Rightarrow$ Minimally wild concealed algebras are not g-tame $\blacksquare$

Incidence algebras of Posets

Thm [Leszczyński '03]

The incidence algebra of a poset is wild iff its universal galois cover contains a minimally wild concealed algebra as a convex subalgebra.

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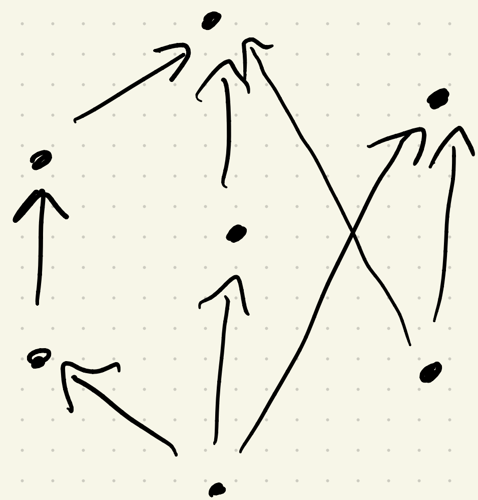
The incidence algebra of a poset is wild iff its universal galois cover contains a minimally wild concealed algebra as a convex subalgebra.

Cor: wild simply connected posets are not g-tame.

Multiply connected posets?



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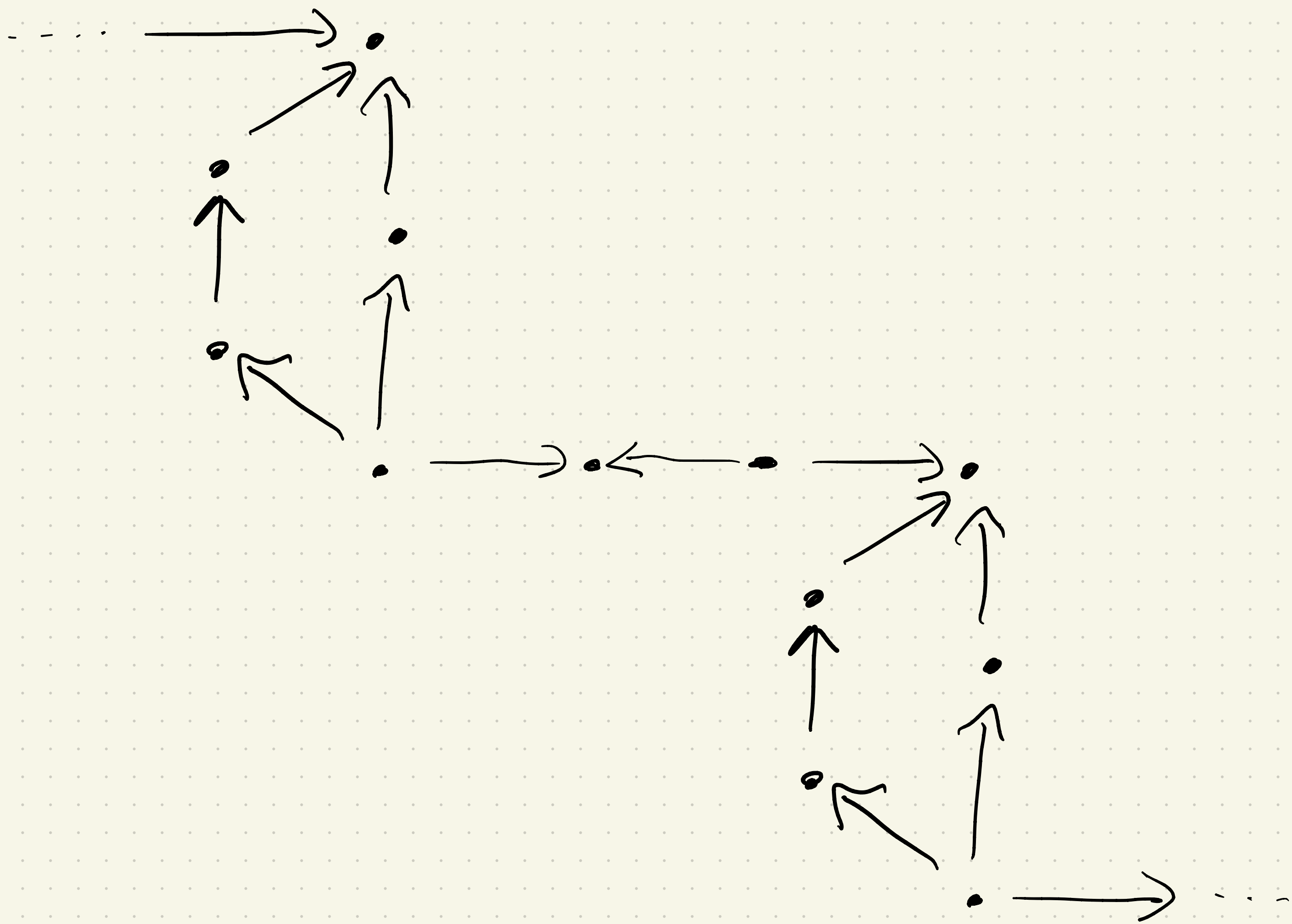

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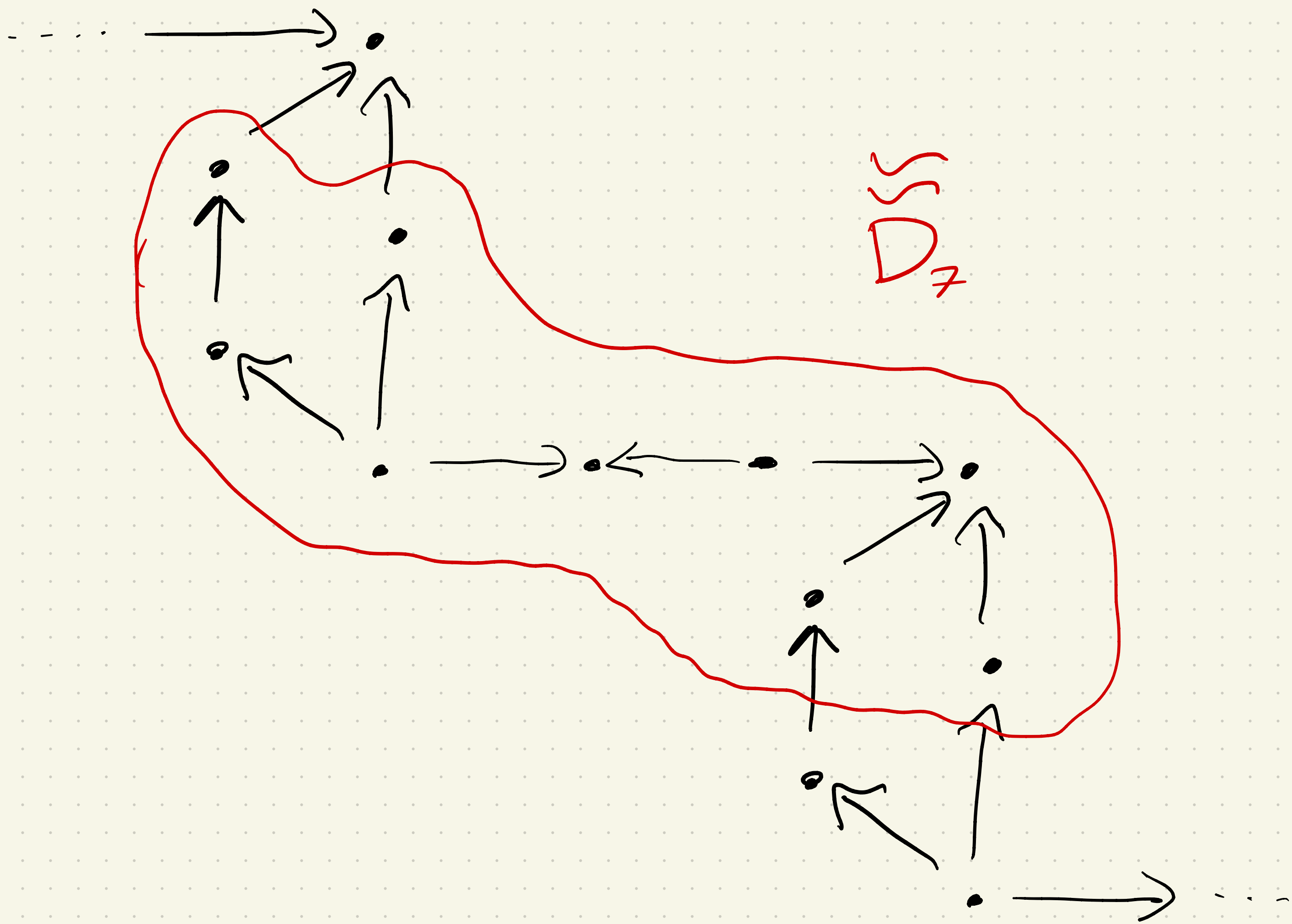

Thank You

for your attention!